The decomposition of level-1 irreducible highest-weight modules with respect to the level-0 actions of the quantum affine algebra

1998 J. Phys. A: Math. Gen. 31 1467

(http://iopscience.iop.org/0305-4470/31/5/015)

J. Phys. A: Math. Gen. **31** (1998) 1467–1485. Printed in the UK PII: S0305-4470(98)86341-5

The decomposition of level-1 irreducible highest-weight modules with respect to the level-0 actions of the quantum affine algebra

Kouichi Takemura†

Research Institute for Mathematical Sciences, Kyoto University, Sakyo–ku, Kyoto 606–01, Japan

Received 28 July 1997

Abstract. The level-1 irreducible highest-weight modules of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_n)$ are decomposed into irreducible components with respect to the level-0 $U'_q(\widehat{\mathfrak{sl}}_n)$-action previously defined in [13]. The components of the decomposition are found to be the so-called tame representations of $U'_q(\widehat{\mathfrak{sl}}_n)$ parametrized by the skew Young diagrams of the border-strip type. This result verifies a recent conjecture due to Kirillov *et al*.

1. Introduction

It is well known that quantum groups often appear as non-Abelian symmetries of one-dimensional exactly solvable models of statistical mechanics. Existence of such symmetries has important implications for physics described by this type of model as well as for the actual choice of a procedure used to obtain exact solution.

An example, much studied recently is provided by the long-range interacting $\mathfrak{sl}_n$-invariant Haldane–Shastry spin chains. In this case the complete non-Abelian symmetry is $Y(\mathfrak{sl}_n)$, the Yangian of $\mathfrak{sl}_n$. After taking an appropriate continuum limit, the space of states of an $\mathfrak{sl}_n$-invariant Haldane–Shastry spin chain is identified with the sum of level-1 irreducible highest weight modules of the affine Lie algebra $\widehat{\mathfrak{sl}}_n$. The Yangian symmetry present in the finite-size model gives rise to $Y(\mathfrak{sl}_n)$-action on each of the level-1 highest weight modules [7]. Explicit expressions for the generators of the $Y(\mathfrak{sl}_n)$-action on a highest weight module of $\widehat{\mathfrak{sl}}_n$ with $n \geqslant 3$ were obtained in [14].

An important problem is to obtain irreducible decomposition of a highest weight $\widehat{\mathfrak{sl}}_n$-module relative to this Yangian action. Each irreducible component gives an eigenspace of the continuum limit of the Hamiltonian. Moreover, the decomposition may be used to obtain novel character formulae for the highest weight module.

The Yangian decomposition in the $\widehat{\mathfrak{sl}}_2$ case was accomplished in [2]. To our knowledge, until now no complete result on the Yangian decomposition for $n \geqslant 3$ has been published. However, recently Kirillov *et al* [10] proposed a remarkable conjecture which purports to describe the combinatorial structure of this decomposition, i.e. $\mathfrak{sl}_n$-characters and equivalence classes (up to certain exterior $Y(\mathfrak{sl}_n)$-automorphisms) of all irreducible components. The distinctive feature of the conjecture is the prediction that all the irreducible components are in the class of the so-called *tame* $Y(\mathfrak{sl}_n)$-modules parametrized by skew Young diagrams of the border-strip type.

† E-mail address: takemura@kurims.kyoto-u.ac.jp

The main result of this paper is a proof of this conjecture. Actually we carry out the proof in a more general, q-deformed, set-up. Instead of decomposing a level-1 highest-weight module of $\widehat{\mathfrak{sl}}_n$ with respect to the Yangian $Y(\mathfrak{sl}_n)$, we decompose a level-1 highest-weight module of $U_q(\widehat{\mathfrak{sl}}_n)$ with respect to the level-0 action of $U'_q(\widehat{\mathfrak{sl}}_n)$. The latter action is a q-deformation of the Yangian. At generic values of the deformation parameter q the decomposition is combinatorially the same as for the Yangian case. We also remark that for $n = 2$ the decomposition we obtain coincides with the previous result of [8]. For details see section 4.3.

Let us now explain features of the method we use to prove the conjecture of [10].

The central role in our approach to the problem of the level-0 $U'_q(\widehat{\mathfrak{sl}}_n)$ decomposition is played by the q-deformed Fock-space module of $U_q(\widehat{\mathfrak{sl}}_n)$ taken in the semi-infinite q-wedge realization due to Kashiwara *et al* [9].

In the papers [19] and [13] it was shown that the q-deformed Fock space module of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_n)$ admits an action of a new remarkable object - the so-called quantum toroidal algebra introduced in [5] and [18] as a q-deformation of the universal central extension of the $\mathfrak{sl}_n$-valued double-loop Lie algebra. The action of the quantum toroidal algebra on the q-Fock space depends on two parameters: the deformation parameter q and an extra parameter p, when values of these parameters are taken to be generic complex numbers, the q-Fock space is known to be irreducible with respect to this action.

The quantum toroidal algebra has two subalgebras, $U'_q(\widehat{\mathfrak{sl}}_n)^{(1)}$ and $U'_q(\widehat{\mathfrak{sl}}_n)^{(2)}$, both isomorphic to $U'_q(\widehat{\mathfrak{sl}}_n)$. Accordingly, the q-Fock space admits two $U'_q(\widehat{\mathfrak{sl}}_n)$-actions.

The first of these actions has level 1, and coincides with the action originally introduced by Hayashi in [6]. The irreducible decomposition of the q-Fock space with respect to this action was given in [9] by using the semi-infinite q-wedge construction due to [15].

The second of the $U'_q(\widehat{\mathfrak{sl}}_n)$ actions has level 0, the irreducible decomposition of the q-Fock space with respect to this action was constructed in [16] at generic values of the parameters q and p.

Kashiwara *et al* [9] have shown, that the level-1 action of $U'_q(\widehat{\mathfrak{sl}}_n)$ on the q-Fock space is centralized by the action of the Heisenberg algebra. In the paper [13] it was proven that the proper ideal of the q-Fock space generated by the negative-frequency part of the Heisenberg algebra is invariant under the action of the quantum toroidal algebra provided the value of the parameter p in the latter is set to be equal to 1. The quotient of the q-Fock space by this ideal is isomorphic to one of the irreducible level-1 highest weight modules of $U_q(\widehat{\mathfrak{sl}}_n)$. As a consequence, each of these modules admits an action of the quantum toroidal algebra.

The corresponding action of the subalgebra $U'_q(\widehat{\mathfrak{sl}}_n)^{(1)}$ is irreducible, it is just the standard level-1 action on the highest-weight irreducible module of $U_q(\widehat{\mathfrak{sl}}_n)$. On the other hand, the action of the subalgebra $U'_q(\widehat{\mathfrak{sl}}_n)^{(2)}$ has level 0 and is completely reducible. The construction of the irreducible decomposition of the level-1 $U_q(\widehat{\mathfrak{sl}}_n)$-modules relative to the level-0 action is the problem which we address in this paper.

2. The actions of the quantum affine algebra $U'_q(\widehat{\mathfrak{sl}}_n)$

2.1. Definition of the quantum affine algebra $U'_q(\widehat{\mathfrak{sl}}_n)$

Definition 1. The quantum affine algebra $U'_q(\widehat{\mathfrak{sl}}_n)$ is the unital associative algebra over $\mathbb{C}$

with generators E_i, F_i, $K_i^{\pm 1}$ ($i \in I := \{0, 1, \ldots, n-1\}$) and the following defining relations:

$$K_i K_i^{-1} = 1 = K_i^{-1} K_i \tag{2.1}$$

$$K_i K_j = K_j K_i \tag{2.2}$$

$$K_i E_j K_i^{-1} = q^{a_{ij}} E_j \tag{2.3}$$

$$K_i F_j K_i^{-1} = q^{-a_{ij}} F_j \tag{2.4}$$

$$[E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}} \tag{2.5}$$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1 - a_{ij} \\ r \end{bmatrix}_q (E_i)^r E_j (E_i)^{1-a_{ij}-r} = 0 \qquad i \neq j \tag{2.6}$$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1 - a_{ij} \\ r \end{bmatrix}_q (F_i)^r F_j (F_i)^{1-a_{ij}-r} = 0 \qquad i \neq j \tag{2.7}$$

where

$$[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}} \qquad \begin{bmatrix} n \\ r \end{bmatrix}_q := \frac{[n]_q [n-1]_q \ldots [n-r+1]_q}{[r]_q [r-1]_q \ldots [1]_q} \tag{2.8}$$

$$a_{ij} = \begin{cases} 2 & (i = j) \\ -1 & (|i - j| = 1, (i, j) = (1, n), (n, 1)) \qquad n \geqslant 3 \\ 0 & (\text{otherwise}) \end{cases} \tag{2.9}$$

$$a_{ij} = \begin{cases} 2(i = j) & n = 2 \\ -2(i \neq j) & n = 2. \end{cases} \tag{2.10}$$

The coproduct Δ is given by

$$\Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i \tag{2.11}$$

$$\Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i \tag{2.12}$$

$$\Delta(K_i) = K_i \otimes K_i. \tag{2.13}$$

We put $c' := K_0 K_1 \ldots K_{n-1}$ in $U_q'(\widehat{\mathfrak{sl}}_n)$, then c' is the central in $U_q'(\widehat{\mathfrak{sl}}_n)$.

2.2. q-wedge product and semi-infinite q-wedge product

The affine Hecke algebra of type $\mathfrak{gl}_N$, $\hat{H}_N(q)$ is a unital associative algebra over $\mathbb{C}[q^{\pm 1}]$ with generators $T_i^{\pm 1}$, $Y_j^{\pm 1}$, $i = 1, 2, \ldots, N-1$, $j = 1, 2, \ldots, N$ and relations

$$T_i T_i^{-1} = T_i^{-1} T_i = 1 \qquad (T_i + 1)(T_i - q^2) = 0$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$$

$$T_i T_j = T_j T_i \qquad \text{if } |j - i| > 1$$

$$Y_i Y_j = Y_j Y_i \qquad T_i^{-1} Y_i T_i^{-1} = q^{-2} Y_{i+1}$$

$$Y_j T_i = T_i Y_j \qquad \text{if } j \neq i, i+1.$$

The subalgebra $H_N(q)$ generated by $T_i^{\pm 1}$ is isomorphic to the Hecke algebra of type $\mathfrak{gl}_N$.

Let $p \in \mathbb{C}^\times$ and consider the following operators in $\mathrm{End}(\mathbb{C}[z_1^{\pm 1}, \ldots, z_N^{\pm 1}])$

$$g_{i,j} = \frac{q^{-1} z_i - q z_j}{z_i - z_j} (K_{i,j} - 1) + q \qquad 1 \leqslant i \neq j \leqslant N$$

$$Y_i^{(N)} = g_{i,i+1}^{-1} K_{i,i+1} \ldots g_{i,N}^{-1} K_{i,N} p^{D_i} K_{1,i} g_{1,i} \ldots K_{i-1,i} g_{i-1,i} \qquad i = 1, 2, \ldots, N$$

where $K_{i,j}$ acts on $\mathbb{C}[z_1^{\pm 1}, \ldots, z_N^{\pm 1}]$ by permuting variables z_i, z_j and p^{D_i} is the difference operator

$$p^{D_i} f(z_1, \ldots, z_i, \ldots, z_N) = f(z_1, \ldots, pz_i, \ldots, z_N) \qquad f \in \mathbb{C}[z_1^{\pm 1}, \ldots, z_N^{\pm 1}].$$

Then the assignment

$$T_i \mapsto \overset{c}{T}_i = -qg_{i,i+1}^{-1} \qquad Y_i \mapsto q^{1-N} Y_i^{(N)} \tag{2.14}$$

defines a right action of $\hat{H}_N(q)$ on $\mathbb{C}[z_1^{\pm 1}, \ldots, z_N^{\pm 1}]$.

The commuting difference operators $Y_1^{(N)}, \ldots, Y_N^{(N)}$ are called Cherednik's operators.

Moreover, the assignment

$$T_i \mapsto \overset{c}{T}_i = -qg_{i,i+1}^{-1} \qquad Y_i \mapsto z_i^{-1}(\text{multiplication}) \tag{2.15}$$

defines another right action of $\hat{H}_N(q)$ on $\mathbb{C}[z_1^{\pm 1}, \ldots, z_N^{\pm 1}]$.

Remark. The actions of $-qg_{i,i+1}^{-1}, q^{1-N}Y_i^{(N)}, z_i^{-1}$ are related to the toroidal Hecke algebra [18] or the double affine Hecke algebra [4].

Let $V = \mathbb{C}^n$, with basis $\{v_1, \ldots, v_n\}$. Then $\otimes^N V$ admits a left $H_N(q)$-action given by

$$T_i \mapsto \overset{s}{T}_i = 1^{\otimes^{i-1}} \otimes \overset{s}{T} \otimes 1^{\otimes^{N-i-1}} \qquad \text{where } \overset{s}{T} \in \mathrm{End}(\otimes^2 V) \tag{2.16}$$

and

$$\overset{s}{T}(v_{\epsilon_1} \otimes v_{\epsilon_2}) = \begin{cases} q^2 v_{\epsilon_1} \otimes v_{\epsilon_2} & \text{if } \epsilon_1 = \epsilon_2 \\ q v_{\epsilon_2} \otimes v_{\epsilon_1} & \text{if } \epsilon_1 < \epsilon_2 \\ q v_{\epsilon_2} \otimes v_{\epsilon_1} + (q^2 - 1) v_{\epsilon_1} \otimes v_{\epsilon_2} & \text{if } \epsilon_1 > \epsilon_2. \end{cases} \tag{2.17}$$

Let $V(z) = \mathbb{C}[z^{\pm 1}] \otimes V$, with basis $\{z^m \otimes v_\epsilon\}$, $m \in \mathbb{Z}$, $\epsilon \in \{1, 2, \ldots, n\}$. Often it will be convenient to set $k = \epsilon - nm$ and $u_k = z^m \otimes v_\epsilon$. Then $\{u_k\}, k \in \mathbb{Z}$ is a basis of $V(z)$. In what follows we will write $z^m v_\epsilon$ as a short-hand for $z^m \otimes v_\epsilon$, and use both notations: u_k and $z^m v_\epsilon$ switching between them according to convenience. The two actions of the Hecke algebra are naturally extended on the tensor product $\mathbb{C}[z_1^{\pm 1}, \ldots, z_N^{\pm 1}] \otimes (\otimes^N V)$ so that $\overset{c}{T}_i$ acts trivially on $\otimes^N V$ and $\overset{s}{T}_i$ acts trivially on $\mathbb{C}[z_1^{\pm 1}, \ldots, z_N^{\pm 1}]$. The vector space $\otimes^N V(z)$ is identified with $\mathbb{C}[z_1^{\pm 1}, \ldots, z_N^{\pm 1}] \otimes (\otimes^N V)$ and the q-wedge product [9] is defined as the following quotient space:

$$\wedge^N V(z) = \otimes^N V(z) \Big/ \sum_{i=1}^{N-1} \mathrm{Ker}(\overset{c}{T}_i + q^2(\overset{s}{T}_i)^{-1}). \tag{2.18}$$

Let $\Lambda : \otimes^N V(z) \to \wedge^N V(z)$ be the quotient map specified by (2.18). The image of a pure tensor $u_{k_1} \otimes u_{k_2} \otimes \cdots \otimes u_{k_N}$ under this map is called a wedge and is denoted by

$$u_{k_1} \wedge u_{k_2} \wedge \ldots \wedge u_{k_N} := \Lambda(u_{k_1} \otimes u_{k_2} \otimes \cdots \otimes u_{k_N}). \tag{2.19}$$

A wedge is normally ordered if $k_1 > k_2 > \cdots > k_N$. In [9] it is proven that normally ordered wedges form a basis in $\wedge^N V(z)$.

Let us now define the semi-infinite q-wedge product $\wedge^{\frac{\infty}{2}} V(z)$ and for any integer M its subspace F_M, following [9].

Let $\otimes^{\frac{\infty}{2}} V(z)$ be the space spanned by the vectors $u_{k_1} \otimes u_{k_2} \otimes \ldots$, $(k_{i+1} = k_i - 1, i \gg 1)$. We define the space $\wedge^{\frac{\infty}{2}} V(z)$ as the quotient of $\otimes^{\frac{\infty}{2}} V(z)$:

$$\wedge^{\frac{\infty}{2}} V(z) := \otimes^{\frac{\infty}{2}} V(z) \Big/ \sum_{i=1}^{\infty} \mathrm{Ker}(\overset{c}{T}_i + q^2 (\overset{s}{T}_i)^{-1}). \tag{2.20}$$

Let $\Lambda : \otimes^{\frac{\infty}{2}} V(z) \to \wedge^{\frac{\infty}{2}} V(z)$ be the quotient map specified by (2.20). The image of a pure tensor $u_{k_1} \otimes u_{k_2} \otimes \ldots$ under this map is called a semi-infinite wedge and is denoted by

$$u_{k_1} \wedge u_{k_2} \wedge \ldots := \Lambda(u_{k_1} \otimes u_{k_2} \otimes \cdots). \tag{2.21}$$

A semi-infinite wedge is normally ordered if $k_1 > k_2 > \cdots$ and $k_{i+1} = k_i - 1 (i \gg 1)$. In [9] it is proven that normally ordered semi-infinite wedges form a basis in $\wedge^{\frac{\infty}{2}} V(z)$.

Let U_M be the subspace of $\otimes^{\frac{\infty}{2}} V(z)$ spanned by the vectors $u_{k_1} \otimes u_{k_2} \otimes \ldots$, $(k_i = M - i + 1, i \gg 1)$. Let F_M be the quotient space of U_M defined by the map (2.21). Then F_M is a subspace of $\wedge^{\frac{\infty}{2}} V(z)$, and the vectors $u_{k_1} \wedge u_{k_2} \wedge \ldots$, $(k_1 > k_2 > \ldots$, $k_i = M - i + 1, i \gg 1)$ form a basis of F_M. We will call the space F_M the q-deformed Fock space.

2.3. Actions of the quantum affine algebra on the q-wedge product

We will define two actions of $U'_q(\widehat{\mathfrak{sl}}_n)$ on the space $\wedge^N V(z)$.

The first one is defined as follows.

$$E_i(m \otimes v) = \sum_{j=1}^{N} m \otimes E_j^{i,i+1} K_{j+1}^i \ldots K_N^i v \tag{2.22}$$

$$F_i(m \otimes v) = \sum_{j=1}^{N} m \otimes (K_1^i)^{-1} \ldots (K_{j-1}^i)^{-1} E_j^{i+1,i} v \tag{2.23}$$

$$K_i(m \otimes v) = m \otimes K_1^i K_2^i \ldots K_N^i v \qquad (i = 1, 2, \ldots, n-1) \tag{2.24}$$

$$E_0(m \otimes v) = \sum_{j=1}^{N} m Y_j^{-1} \otimes E_j^{n,1} K_{j+1}^0 \ldots K_N^0 v \tag{2.25}$$

$$F_0(m \otimes v) = \sum_{j=1}^{N} m Y_j \otimes (K_1^0)^{-1} \ldots (K_{j-1}^0)^{-1} E_j^{1,n} v \tag{2.26}$$

$$K_0 = (K_1 K_2 \ldots K_{n-1})^{-1}. \tag{2.27}$$

Here $E_j^{i,k} = 1^{\otimes j-1} \otimes E^{i,k} \otimes 1^{\otimes N-j}$, where $E^{i,k} \in \mathrm{End}(V)$ is the matrix unit in the basis $v_1, \ldots, v_n$, and $K_j^i = q^{E_j^{i,i} - E_j^{i+1,i+1}}$, $K_j^0 = (K_j^1 K_j^2 \ldots K_j^{n-1})^{-1}$.

We will denote this action by $U_0^{(N)}$. Note that it is well defined on the quotient space $\wedge^N V(z)$ in view of the relations of the affine Hecke algebra.

The second one is defined as follows.

$$E_0(m \otimes v) = \sum_{j=1}^{N} m z_j \otimes E_j^{n,1} K_{j+1}^0 \ldots K_N^0 v \tag{2.28}$$

$$F_0(m \otimes v) = \sum_{j=1}^{N} m z_j^{-1} \otimes (K_1^0)^{-1} \ldots (K_{j-1}^0)^{-1} E_j^{1,n} v. \tag{2.29}$$

The actions of other generators are the same as in (2.22)–(2.24), (2.27).

We will denote this action by $U_1^{(N)}$. Again, this action is well defined on the quotient space $\wedge^N V(z)$ in view of the relations of the affine Hecke algebra.

2.4. Level-0 action of the quantum affine algebra on the q-deformed Fock space

We will define a level-0 action of $U'_q(\widehat{\mathfrak{sl}}_n)$ on $F_M (M \in \mathbb{Z})$ following the paper [16, 13].

Let $\boldsymbol{e} := (\epsilon_1, \epsilon_2, \ldots, \epsilon_N)$ where $\epsilon_i \in \{1, 2, \ldots, n\}$. For a sequence $\boldsymbol{e}$ we set

$$\boldsymbol{v_e} := v_{\epsilon_1} \otimes v_{\epsilon_2} \otimes \cdots \otimes v_{\epsilon_N} \qquad (\in \otimes^N \mathbb{C}^n). \tag{2.30}$$

A sequence $\boldsymbol{m} := (m_1, m_2, \ldots, m_N)$ from $\mathbb{Z}^N$ is called n-strict if it contains no more than n equal elements of any given value. Let us define the sets $\mathcal{M}_N^n$ and $\mathcal{E}(\boldsymbol{m})$ by

$$\mathcal{M}_N^n := \{\boldsymbol{m} = (m_1, m_2, \ldots, m_N) \in \mathbb{Z}^N | m_1 \leqslant m_2 \leqslant \cdots \leqslant m_N, \boldsymbol{m} \text{ is } n\text{-strict}\} \tag{2.31}$$

and for $\boldsymbol{m} \in \mathcal{M}_N^n$

$$\mathcal{E}(\boldsymbol{m}) := \{\boldsymbol{e} = (\epsilon_1, \epsilon_2, \ldots, \epsilon_N) \in \{1, 2, \ldots, n\}^N | \epsilon_i > \epsilon_{i+1} \qquad \forall \boldsymbol{i} \text{ s.t. } m_i = m_{i+1}\}. \tag{2.32}$$

In these notations the set

$$\{w(\boldsymbol{m}, \boldsymbol{e}) := \Lambda(\boldsymbol{z^m} \otimes \boldsymbol{v_e}) = z^{m_1} v_{\epsilon_1} \wedge z^{m_2} v_{\epsilon_2} \wedge \ldots \wedge z^{m_N} v_{\epsilon_N} | \boldsymbol{m} \in \mathcal{M}_N^n, \boldsymbol{e} \in \mathcal{E}(\boldsymbol{m})\} \tag{2.33}$$

is nothing but the base of the normally ordered wedges in $\wedge^N V(z)$. We will use the notation $w(\boldsymbol{m}, \boldsymbol{e})$ *exclusively* for normally ordered wedges.

Similarly for a semi-infinite wedge $w = u_{k_1} \wedge u_{k_2} \wedge \ldots = z^{m_1} v_{\epsilon_1} \wedge z^{m_2} v_{\epsilon_2} \wedge \ldots$, such that $w \in F_M$, the semi-infinite sequences $\boldsymbol{m} = (m_1, m_2, \ldots)$ and $\boldsymbol{e} = (\epsilon_1, \epsilon_2, \ldots)$ are defined by $k_i = \epsilon_i - n m_i$, $\epsilon_i \in \{1, 2, \ldots, n\}$, $m_i \in \mathbb{Z}$. In particular the $\boldsymbol{m}$- and $\boldsymbol{e}$- sequences of the vacuum vector in F_M will be denoted by $\boldsymbol{m}^0$ and $\boldsymbol{e}^0$:

$$|M\rangle = u_M \wedge u_{M-1} \wedge u_{M-2} \wedge \ldots = z^{m_1^0} v_{\epsilon_1^0} \wedge z^{m_2^0} v_{\epsilon_2^0} \wedge z^{m_3^0} v_{\epsilon_3^0} \wedge \ldots. \tag{2.34}$$

The Fock space F_M is $\mathbb{Z}_{\geqslant 0}$-graded. For any semi-infinite wedge $w = u_{k_1} \wedge u_{k_2} \wedge \ldots = z^{m_1} v_{\epsilon_1} \wedge z^{m_2} v_{\epsilon_2} \wedge \ldots \in F_M$ the degree $|w|$ is defined by

$$|w| = \sum_{i \geqslant 1} m_i^0 - m_i. \tag{2.35}$$

Let us denote by $F_M^k \subset F_M$ the homogeneous component of degree k.

We will define a level-0 action of $U'_q(\widehat{\mathfrak{sl}}_n)$ on the Fock space F_M in such a way that each homogeneous component F_M^k will be invariant with respect to this action. Throughout this section we fix an integer M and $s \in \{0, 1, 2, \ldots, n-1\}$ such that $M = s \bmod n$.

Let l be a non-negative integer and define $V_M^{s+nl} \subset \wedge^{s+nl} V(z)$ as follows:

$$V_M^{s+nl} = \bigoplus_{\substack{\boldsymbol{m} \in \mathcal{M}_{s+nl}^n, \boldsymbol{e} \in \mathcal{E}(\boldsymbol{m}) \\ m_{s+nl} \leqslant m_{s+nl}^0}} \mathbb{C} w(\boldsymbol{m}, \boldsymbol{e}). \tag{2.36}$$

The vector space V_M^{s+nl} has a grading similar to the grading of the Fock space F_M. In this case the degree $|w|$ of a wedge $w = u_{k_1} \wedge u_{k_2} \wedge \ldots \wedge u_{k_{s+nl}} = z^{m_1} v_{\epsilon_1} \wedge z^{m_2} v_{\epsilon_2} \wedge \ldots \wedge z^{m_{s+nl}} v_{\epsilon_{s+nl}} \in V_M^{s+nl}$ is defined by

$$|w| = \sum_{i=1}^{s+nl} m_i^0 - m_i. \tag{2.37}$$

The degree is a non-negative integer, and for $k \in \mathbb{Z}_{\geqslant 0}$ we denote by $V_M^{s+nl,k}$ the homogeneous component of degree k.

The following result is contained in the paper [16]:

Proposition 1. For each $k \in \mathbb{Z}_{\geqslant 0}$ the homogeneous component $V_M^{s+nl,k} \subset \wedge^{s+nl} V(z)$ is invariant under the $U'_q(\widehat{\mathfrak{sl}}_n)$-action $U_0^{(s+nl)}$ defined in section 2.3.

We have $|\overline{\rho}_l^{M,k}(w)| = |w|$ and hence $\overline{\rho}_l^{M,k} : V_M^{s+nl,k} \to F_M^k$ for all $k \in \mathbb{Z}_{\geqslant 0}$. In the paper [13] the following propositions are shown.

Proposition 2. When $l \geqslant k$ the map $\overline{\rho}_l^{M,k}$ is an isomorphism of vector spaces.

Proposition 3. For each triple of non-negative integers k, l, m such that $k \leqslant l < m$ the map $\overline{\rho}_{l,m}^{M,k} : V_M^{s+nl,k} \to V_M^{s+nm,k}$, defined for any $w \in V_M^{s+nl,k}$ by

$$\overline{\rho}_{l,m}^{M,k}(w) = w \wedge u_{M-s-nl} \wedge u_{M-s-nl-1} \wedge \ldots \wedge u_{M-s-nm+1} \tag{2.38}$$

is an isomorphism of the $U_q'(\widehat{\mathfrak{sl}}_n)$-modules.

We define on the vector space F_M^k a level-0 action of $U_q'(\widehat{\mathfrak{sl}}_n)$ by using propositions 2 and 3.

Definition 2. The vector space F_M^k is a level-0 module of $U_q'(\widehat{\mathfrak{sl}}_n)$ with the action U_0 defined by

$$U_0 = \overline{\rho}_l^{M,k} U_0^{(s+nl)} \overline{\rho}_l^{M,k^{-1}} \qquad \text{where } l \geqslant k. \tag{2.39}$$

This definition does not depend on the choice of l as long as l is greater or equal to k. Since we have

$$F_M = \bigoplus_{k \geqslant 0} F_M^k \tag{2.40}$$

the level-0 action U_0 extends to the entire Fock space F_M.

2.5. Level-1 action of the quantum affine algebra on the q-deformed Fock space

In this section we review the level-1 action of $U_q'(\widehat{\mathfrak{sl}}_n)$ on the Fock space F_M [9].

First we define the action of $U_q'(\widehat{\mathfrak{sl}}_n)$ (generated by $E_i, F_i, K_i, i = 0, \ldots, n-1$) on the vector $|M'\rangle$ as follows.

$$E_i|M'\rangle = 0 \tag{2.41}$$

$$F_i|M'\rangle = \begin{cases} u_{M'+1} \wedge u_{M'-1} \wedge u_{M'-2} \wedge \ldots & \text{if } i \equiv M' \bmod n \\ 0 & \text{otherwise} \end{cases} \tag{2.42}$$

$$K_i|M'\rangle = \begin{cases} q|M'\rangle & \text{if } i \equiv M' \bmod n \\ |M'\rangle & \text{otherwise.} \end{cases} \tag{2.43}$$

For every element $v \in F_M$, there exists N such that

$$v = v^{(N)} \wedge |M-N\rangle \qquad v^{(N)} \in \wedge^N V(z). \tag{2.44}$$

We define the actions of $E_i, F_i, K_i, i = 0, \ldots, n-1$ on the vector v as follows.

$$E_i v := E_i v^{(N)} \wedge K_i|M-N\rangle + v^{(N)} \wedge E_i|M-N\rangle \tag{2.45}$$

$$F_i v := F_i v^{(N)} \wedge |M-N\rangle + K_i^{-1} v^{(N)} \wedge F_i M-N\rangle \tag{2.46}$$

$$K_i v := K_i v^{(N)} \wedge K_i|M-N\rangle. \tag{2.47}$$

The actions of E_i, F_i, K_i, $i = 0, \ldots, n-1$ on $v^{(N)}$ are determined in section 2.3. The definition of the actions on v does not depend on N and is well defined, and we can easily check that the $U_q'(\widehat{\mathfrak{sl}}_n)$-module defined in this section is level-1. We will use the notation U_1 for this $U_q'(\widehat{\mathfrak{sl}}_n)$-action on the Fock space.

Remark. The two actions U_0 and U_1 appear as the representations of the subalgebras of the quantum toroidal algebra. For details, see [13].

2.6. The $p = 1$ case

In the paper [9] it was demonstrated that the Fock space F_M admits an action of the Heisenberg algebra H which commutes with the level-1 action U_1 of the algebra $U_q'(\widehat{\mathfrak{sl}}_n)$. The Heisenberg algebra is a unital $\mathbb{C}$-algebra generated by elements 1, B_a with $a \in \mathbb{Z}_{\neq 0}$ which are subject to relations

$$[B_a, B_b] = \delta_{a+b,0}\, a \frac{1-q^{2na}}{1-q^{2a}}. \tag{2.48}$$

The Fock space F_M is an H-module with the action of the generators given by [9]

$$B_a = \sum_{i=1}^{\infty} z_i^a. \tag{2.49}$$

Let $\mathbb{C}[H_-]$ be the Fock space of H, i.e. $\mathbb{C}[H_-] = \mathbb{C}[B_{-1}, B_{-2}, \ldots,]$. The element $B_{-a} (a = 1, 2, \ldots)$ acts on $\mathbb{C}[H_-]$ by multiplication. The action of $B_a (a = 1, 2, \ldots)$ is given by (2.48) together with the relation

$$B_a \cdot 1 = 0 \qquad \text{for } a \geqslant 1. \tag{2.50}$$

Let $\Lambda_i (i \in \{0, 1, \ldots, n-1\})$ be the fundamental weights of $\widehat{\mathfrak{sl}}_n'$ and let $V(\Lambda_i)$ be the irreducible (level-1) highest weight module of $U_q'(\widehat{\mathfrak{sl}}_n)$ with highest weight vector V_{Λ_i} and highest weight Λ_i.

The following results are proven in [9].

- The action of the Heisenberg algebra on F_M and the action U_1 of $U_q'(\widehat{\mathfrak{sl}}_n)$ commute.
- There is an isomorphism

$$\iota_M : F_M \cong V(\Lambda_i) \otimes \mathbb{C}[H_-] \qquad (M = i \bmod n) \tag{2.51}$$

of $U_q'(\widehat{\mathfrak{sl}}_n) \otimes H$-modules normalized so that $\iota_M(|M\rangle) = V(\Lambda_i) \otimes 1$.

In general the level-0 $U_q'(\widehat{\mathfrak{sl}}_n)$-action U_0 does not commute with the Heisenberg algebra. However, if we choose the parameter p in U_0 in a special way, then U_0 commute with the negative frequency part of H. Precisely, we have the following proposition, proved in [13].

Proposition 4. At $p = 1$ we have

$$[U_0, H_-] = 0. \tag{2.52}$$

Let H_-' be the non-unital subalgebra in H generated by $B_{-1}, B_{-2}, \ldots$. Proposition 4 allows us to define a level-0 $U_q'(\widehat{\mathfrak{sl}}_n)$-module structure on the irreducible level-1 module $V(\Lambda_i) (i \in \{0, 1, \ldots, n-1\})$. Indeed from this proposition it follows that the subspace

$$H_-' F_M \subset F_M \tag{2.53}$$

is invariant with respect to the action U_0 at $p = 1$ and therefore a level-0 action of $U_q'(\widehat{\mathfrak{sl}}_n)$ is defined on the quotient space

$$F_M/(H_-' F_M) \tag{2.54}$$

which in view of (2.51) is isomorphic to $V(\Lambda_i)$ with $i \equiv M \bmod n$.

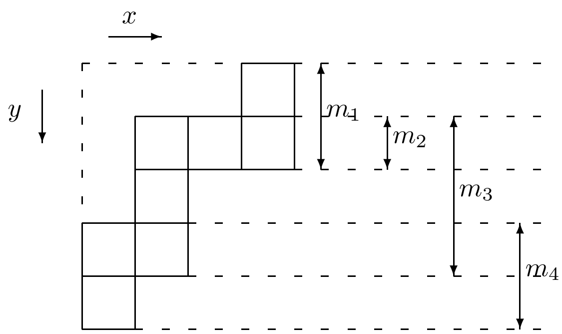

Figure 1. $\lambda = (4, 4, 2, 2, 1)$, $\mu = (3, 1, 1)$.

3. Skew Young diagrams and the level-0 representations of $U'_q(\widehat{\mathfrak{sl}}_n)$

3.1. Skew Young diagrams

Let us recall, following the book [12], the definitions of the skew (Young) diagrams, their semi-standard tableaux (SST) and the associated skew Schur functions.

Let λ, μ be partitions i.e. sequences of non-negative integers. We assume $\lambda_i \geqslant \mu_i$ for all possible i, and if $\mu_j < i \leqslant \lambda_j$ then we draw a square whose edges are $(i-1, j-1)$, $(i-1, j)$, (i, j) and $(i, j-1)$. (For example, see figure 1.) This diagram is called a skew (Young) diagram and is denoted as $\lambda \setminus \mu$. We define the degree of the skew Young diagram $\lambda \setminus \mu$ as $|\lambda \setminus \mu| = \sum_i (\lambda_i - \mu_i)$.

A skew diagram is called a border strip if it is connected and contains no 2×2 blocks of boxes. Let $\langle m_1, \ldots, m_r \rangle$ denote the border strip of r columns such that the length of ith column (from the right) is m_i (figure 1).

A SST of the skew diagram $\lambda \setminus \mu$ is obtained by inscribing integers $1, 2, \ldots, n$ in each square of the skew diagram. The rule of the SST is as follows. The numbers are strictly increasing along the column and weakly increasing along the row. For each SST T, let $n_i(T)$ be the multiplicity of i in T.

Definition 3. For each skew diagram $\lambda \setminus \mu$, the skew Schur function $s_{\lambda\setminus\mu}$ is defined as follows

$$s_{\lambda\setminus\mu}(z) = \sum_T z_1^{n_1(T)} z_2^{n_2(T)} \ldots z_N^{n_N(T)}. \tag{3.1}$$

Here the summation is over the set of SST of the skew diagram $\lambda \setminus \mu$.

3.2. The level-0 representations of $U'_q(\widehat{\mathfrak{sl}}_n)$ associated with the skew diagrams

Fix a skew diagram $\lambda\setminus\mu$ of the border-strip type and degree N. We put a number $(1, \ldots, N)$ on each box such that if $l > k$ then $x_l > x_k$ or $(x_l = x_k$ and $y_l > y_k)$, where (x, y) is a box contained in the skew diagram, and set $a_l = -2x_l + 2y_l + a$ (a is fixed) (figure 2).

On the space $\otimes^N V$, we define the evaluation action $\pi^{(N)}_{a_1,\ldots,a_N}$ of $U'_q(\widehat{\mathfrak{sl}}_n)$

$$\pi^{(N)}_{a_1,\ldots,a_N}(E_i) = \sum_{j=1}^{N} E_j^{i,i+1} K_{j+1}^i \ldots K_N^i \tag{3.2}$$

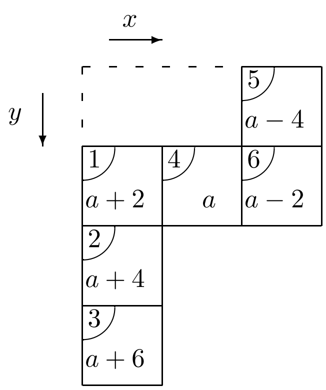

Figure 2. The case $\langle 2, 1, 3\rangle$.

$$\pi^{(N)}_{a_1,\ldots,a_N}(F_i) = \sum_{j=1}^{N} (K_1^i)^{-1}\ldots(K_{j-1}^i)^{-1}E_j^{i+1,i} \tag{3.3}$$

$$\pi^{(N)}_{a_1,\ldots,a_N}(K_i) = K_1^iK_2^i\ldots K_N^i \qquad (i=1,2,\ldots,n-1) \tag{3.4}$$

$$\pi^{(N)}_{a_1,\ldots,a_N}(E_0) = \sum_{j=1}^{N} q^{a_j}E_j^{n,1}K_{j+1}^0\ldots K_N^0 \tag{3.5}$$

$$\pi^{(N)}_{a_1,\ldots,a_N}(F_0) = \sum_{j=1}^{N} q^{-a_j}(K_1^0)^{-1}\ldots(K_{j-1}^0)^{-1}E_j^{1,n} \tag{3.6}$$

$$\pi^{(N)}_{a_1,\ldots,a_N}(K_0) = (\pi^{(N)}_{a_1,\ldots,a_N}(K_1K_2\cdots K_{n-1}))^{-1} \tag{3.7}$$

and on the same space, we consider the following operators:

$$R_{i,j}(x) = \frac{xS_{i,j}^{-1} - S_{i,j}}{x-1}P_{i,j} \qquad \check{R}_{i,j}(x) = \frac{xS_{i,j}^{-1} - S_{i,j}}{x-1} \tag{3.8}$$

where $P_{i,j}(\cdots\otimes \overset{i}{u} \otimes\cdots\otimes \overset{j}{v}\otimes\cdots) = \cdots\otimes \overset{i}{v}\otimes\cdots\otimes\overset{j}{u}\otimes\cdots$. We define

$$R_{\lambda\backslash\mu} = \prod_{1\leqslant i<j\leqslant N} R_{i,j}(q^{a_i-a_j}) \tag{3.9}$$

$$\bar{R}_{\lambda\backslash\mu} = \prod_{1\leqslant i<j\leqslant N} R_{j,i}(q^{a_i-a_j}) \tag{3.10}$$

$$\check{R}_{\lambda\backslash\mu} = \prod_{1\leqslant i<j\leqslant N} \check{R}_{N+i-j,N+i-j+1}(q^{a_i-a_j}) \tag{3.11}$$

where (i, j) is on the right to (i', j') in the product if $i < i'$ or ($j < j'$ and $i = i'$). As a special case of [3] proposition 1.5, we have the following proposition.

Proposition 5 ([3]). The subspace $\mathrm{Im}\, R_{\lambda\backslash\mu}(\otimes^N V)(=\ \mathrm{Im}\, \check{R}_{\lambda\backslash\mu}(\otimes^N V))$ with the action $\pi^{(N)}_{a_1,\ldots,a_N}$ is an irreducible $U_q'(\widehat{\mathfrak{sl}}_n)$-module, and the map $\check{R}_{\lambda\backslash\mu}$: $(\pi^{(N)}_{a_N,\ldots,a_1}, \otimes^N V/\ \mathrm{Ker}\ \bar{R}_{\lambda\backslash\mu}) = (\pi^{(N)}_{a_N,\ldots,a_1}, \otimes^N V/\ \mathrm{Ker}\ \check{R}_{\lambda\backslash\mu})) \to (\pi^{(N)}_{a_1,\ldots,a_N}, \mathrm{Im}\ \check{R}_{\lambda\backslash\mu})$ is an isomorphism of the $U_q'(\widehat{\mathfrak{sl}}_n)$-modules.

Remark. In [3], this proposition is proved in the $U'_q(\widehat{\mathfrak{gl}}_n)$-module case and the normalizations of q and x are different. The irreducibility as the $U'_q(\widehat{\mathfrak{sl}}_n)$-module follows from the result of [1].

3.3. Character formulae

Let $\overline{\Lambda}_i(i = 1, \ldots, n-1)$ be the fundamental weights of $\mathfrak{sl}_n$ and let $\epsilon_i = \overline{\Lambda}_i - \overline{\Lambda}_{i-1}(i = 1, \ldots, n)$ with $\overline{\Lambda}_0 = \overline{\Lambda}_n := 0$.

The subalgebra of $U'_q(\widehat{\mathfrak{sl}}_n)$ generated by $E_i, F_i, K_i^{\pm}(i = 1, \ldots n-1)$ is isomorphic to the algebra $U_q(\mathfrak{sl}_n)$. In the paper [10] the $\mathfrak{sl}_n$-character of the irreducible $Y(\mathfrak{sl}_n)$-representation associated with a skew diagram was shown to be given by the corresponding skew Schur function. This result is immediately generalized to the q-deformed situation. Precisely we have the following proposition.

Proposition 6. [10]. The skew Schur function $s_{\lambda\setminus\mu}(z)$ where $z_i = e^{\epsilon_i}$ is equal to the $U_q(\mathfrak{sl}_n)$-character of the irreducible $U'_q(\widehat{\mathfrak{sl}}_n)$-module described by proposition 5.

As a corollary we obtain the following.

Corollary 1. The dimension of the space $\mathrm{Im}\, R_{\lambda\setminus\mu} \subset (\otimes^N V)$ is equal to the total number of the SST of the skew diagram $\lambda \setminus \mu$.

Let $V(\Lambda_k)$ be the level-1 irreducible module of $U_q(\widehat{\mathfrak{sl}}_n)$ whose highest weight is the kth fundamental weight Λ_k of $\widehat{\mathfrak{sl}}'_n$. We set $\mathrm{ch}(V(\Lambda_k)) = \sum_{i,\lambda}(\dim V_{\lambda,i})e^{\lambda}q^i$, where $V_{\lambda,i}$ is the weight subspace with $U_q(\mathfrak{sl}_n)$-weight λ and homogeneous degree i. The following proposition is proved in [10].

Proposition 7. [10] Setting $z_i = e^{\epsilon_i}$ we have

$$\mathrm{ch}(V(\Lambda_k)) = q^{\frac{1-n}{24} - \frac{k(n-k)}{2n}} \sum_{\substack{\theta \in BS \\ |\theta| \equiv k \bmod n}} q^{\frac{1}{2n}|\theta|(n-|\theta|)+t(\theta)} s_\theta(z) \tag{3.12}$$

where BS is the set of all the border strips $\theta = \langle m_1, \ldots, m_r \rangle$ and $t(\theta) = \sum_{i=1}^{r-1}(r-i)m_i$ with $m_r < n$.

Note that if $m_i > n$ for some i, then the skew Schur function s_θ is equal to 0, moreover, for the border strip of the form $\theta_l = \langle m_1, \ldots, m_r, \overbrace{n, \ldots n}^{l} \rangle$ the number $\frac{1}{2n}|\theta_l|(n-|\theta_l|)+t(\theta_l)$ does not depend on l.

4. Non-symmetric Macdonald polynomials and the decomposition

4.1. Non-symmetric Macdonald polynomials

We will define the non-symmetric Macdonald polynomials as the joint eigenfunctions of the Cherednik's operators $Y_i^{(N)}(i = 1, \ldots, N)$ [17, 16]. It will be convenient for our purposes to label these polynomials by the set of pairs (λ, σ) which we now describe.

Let $\tilde{\mathcal{M}}_N$ be the a set of all non-decreasing sequences of integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N)$ and let $\tilde{\mathcal{M}}_N^n$ be the subset of $\tilde{\mathcal{M}}_N$ which consists of all n-strict non-decreasing sequences (cf section 2.4). For each $\lambda \in \tilde{\mathcal{M}}_N$ we set $|\lambda| := \sum_{i=1}^{N} \lambda_i$. For $\lambda, \mu \in \tilde{\mathcal{M}}_N$ such that $|\lambda| = |\mu|$ we define the dominance (partial) ordering:

$$\lambda \succeq \mu \Leftrightarrow \sum_{j=1}^{i} \lambda_j \geqslant \sum_{j=1}^{i} \mu_j (i \in \{1, 2, \ldots, N\}). \tag{4.1}$$

Let $S^\lambda \subset \mathfrak{S}_N$ be the set of elements σ such that if $\lambda_{\sigma(i)} = \lambda_{\sigma(j)}$ and $\sigma(i) < \sigma(j)$ then $i < j$. We define the total ordering on S^λ:

$$\sigma \succ \sigma' \Leftrightarrow \text{the last non-zero element of } (\lambda_{\sigma(i)} - \lambda_{\sigma'(i)})_{i=1}^{N} \text{ is } < 0. \tag{4.2}$$

Then the following properties are satisfied. (In what follows the $\sigma(i, i+1)$ denotes the composition of σ and a transposition $(i, i+1)$.)

(a) S^λ has the unique minimal element with respect to the ordering (4.2). We denote this element by min. Note that the one we want has $\lambda_{\min(i)} \leqslant \lambda_{\min(i+1)}$ $(i = 1, \ldots, N-1)$.

(b) S^λ is connected, i.e. for any $\sigma \in S^\lambda$, there exist $i_1, \ldots, i_r$ such that if we put $\sigma_l = \sigma(i_1, i_1+1) \ldots (i_l, i_l+1)$ then $\sigma_r = \min$, $\sigma_l \in S^\lambda$, $\sigma_l \succ \sigma_{l+1} (l = 1, \ldots, r)$.

(c) Suppose $\sigma \in S^\lambda$, then $\sigma(i, i+1) \in S^\lambda \Leftrightarrow \lambda_{\sigma(i)} \neq \lambda_{\sigma(i+1)}$.

(d) If $\lambda_{\sigma(i)} > \lambda_{\sigma(i+1)}$ and $\sigma \in S^\lambda$ then $\sigma \succ \sigma(i, i+1)$.

We define the partial ordering of the set $\{(\lambda, \sigma) | \lambda \in \tilde{\mathcal{M}}_N, \sigma \in S^\lambda\}$:

$$(\lambda, \sigma) \succ (\tilde{\lambda}, \tilde{\sigma}) \Leftrightarrow |\lambda| = |\tilde{\lambda}| \text{ and } \begin{cases} \lambda \succ \tilde{\lambda} \\ \lambda = \tilde{\lambda}, \sigma \succ \tilde{\sigma}. \end{cases} \tag{4.3}$$

Then $Y_i^{(N)}$ act triangularly on $\mathbb{C}[z_1^{\pm 1}, \ldots, z_N^{\pm 1}]$ with respect to this ordering [17]:

$$Y_i^{(N)} z^{\lambda^\sigma} = \xi_i^\lambda(\sigma) z^{\lambda^\sigma} + \text{'lower terms'} \tag{4.4}$$

$$\xi_i^\lambda(\sigma) = p^{\lambda_{\sigma(i)}} q^{2\sigma(i)-N-1} \qquad (\sigma \in S^\lambda). \tag{4.5}$$

In the above notation, we identify the ordering of monomials $z^{\lambda^\sigma} := z_1^{\lambda_{\sigma(1)}} z_2^{\lambda_{\sigma(2)}} \ldots z_N^{\lambda_{\sigma(N)}}$ with the ordering on the set of pairs (λ, σ).

For generic q and p the pair (λ, σ) is uniquely determined from the ordered set $(\xi_1^\lambda(\sigma), \xi_2^\lambda(\sigma), \ldots \xi_N^\lambda(\sigma))$:

$$(\lambda, \sigma) \neq (\tilde{\lambda}, \tilde{\sigma}) \Leftrightarrow (\xi_1^\lambda(\sigma), \xi_2^\lambda(\sigma), \ldots \xi_N^\lambda(\sigma)) \neq (\xi_1^{\tilde{\lambda}}(\tilde{\sigma}), \xi_2^{\tilde{\lambda}}(\tilde{\sigma}), \ldots \xi_N^{\tilde{\lambda}}(\tilde{\sigma})). \tag{4.6}$$

Therefore one can simultaneously diagonalize the operators $Y_i^{(N)} (1 \leqslant i \leqslant N)$.

$$Y_i^{(N)} \Phi_\sigma^\lambda(z) = \xi_i^\lambda(\sigma) \Phi_\sigma^\lambda(z) \qquad \Phi_\sigma^\lambda(z) = z^{\lambda^\sigma} + \text{'lower terms'}. \tag{4.7}$$

The Laurent polynomial $\Phi_\sigma^\lambda(z)$ is known as the non-symmetric Macdonald polynomial.

The action of $g_{i,i+1}$ on the non-symmetric Macdonald polynomial is as follows [17].

$$g_{i,i+1} \Phi_\sigma^\lambda(z) = A_i(\sigma) \Phi_\sigma^\lambda(z) + B_i(\sigma) \Phi_{\sigma(i,i+1)}^\lambda(z) \tag{4.8}$$

$$A_i(\sigma) := \frac{(q - q^{-1})x}{x - 1} \qquad B_i(\sigma) := \begin{cases} q^{-1}\{x\} & (\lambda_{\sigma(i)} > \lambda_{\sigma(i+1)}) \\ 0 & (\lambda_{\sigma(i)} = \lambda_{\sigma(i+1)}) \\ q^{-1} & (\lambda_{\sigma(i)} < \lambda_{\sigma(i+1)}) \end{cases} \tag{4.9}$$

$$\{x\} := \frac{(x - q^2)(q^2 x - 1)}{(x - 1)^2} \qquad x := \frac{\xi_{i+1}^\lambda(\sigma)}{\xi_i^\lambda(\sigma)}. \tag{4.10}$$

The case $p = 1$ is not generic. However, from the results of [11] it follows that the coefficients of $\Phi_\sigma^\lambda(z)$ have no poles at $p = 1$. Therefore the non-symmetric Macdonald polynomials $\Phi^{\lambda}_{\sigma}(z)$ are still well defined at $p = 1$ and the formulae (4.7)–(4.10) are still satisfied.

In what follows we will let $\tilde{\Phi}^{\lambda}_{\sigma}(z)$ denote the non-symmetric Macdonald polynomial at $p = 1$. In virtue of the triangularity (4.4) the non-symmetric Macdonald polynomials $\tilde{\Phi}^{\lambda}_{\sigma}(z)(\lambda \in \tilde{\mathcal{M}}_N, \sigma \in S^{\lambda})$ form a base of $\mathbb{C}[z_1^{\pm 1}, \ldots, z_N^{\pm 1}]$. We put

$$E^{\lambda} = \bigoplus_{\sigma \in S^{\lambda}} \mathbb{C}\tilde{\Phi}^{\lambda}_{\sigma}(z). \tag{4.11}$$

Then $\mathbb{C}[z_1^{\pm 1}, \ldots, z_N^{\pm 1}] = \oplus_{\lambda} E^{\lambda}$. In section 4.2 we will use the following lemma.

Lemma 1. Let $e_{-k} = \sum_{1 \leqslant n_1 < \ldots < n_k \leqslant N} z_{n_1}^{-1} \ldots z_{n_k}^{-1}$. Suppose that $\lambda \in \tilde{\mathcal{M}}_N$ satisfies $\lambda_i - \lambda_{i+1} =$ 0 or 1. Then we have

$$e_{-k}\tilde{\Phi}^{\lambda}_{\varsigma}(z) = \tilde{\Phi}^{\tilde{\lambda}}_{\varsigma}(z). \tag{4.12}$$

Here $\tilde{\lambda} = (\lambda_1, \ldots, \lambda_{N-k}, \ldots, \lambda_{N-k+1} - 1, \ldots, \lambda_N - 1)$ and $\varsigma (\in S^{\lambda}, S^{\tilde{\lambda}})$ is the minimal element of S^{λ}.

Proof. By the triangularity of the non-symmetric Macdonald polynomial (4.4), we have

$$\begin{aligned} e_{-k}\tilde{\Phi}^{\lambda}_{\varsigma}(z) &= e_{-k}\left(z^{\lambda^{\varsigma}} + \sum_{\mu \prec \lambda, \sigma \in S^{\mu}} c_{\mu,\sigma} z^{\mu^{\sigma}}\right) = z^{\tilde{\lambda}^{\varsigma}} + \sum_{(\mu,\sigma) \prec (\tilde{\lambda}, \varsigma)} c'_{\mu,\sigma} z^{\mu^{\sigma}} \\ &= \tilde{\Phi}^{\tilde{\lambda}}_{\varsigma}(z) + \sum_{(\mu,\sigma) \prec (\tilde{\lambda}, \varsigma)} c''_{\mu,\sigma} \tilde{\Phi}^{\mu}_{\sigma}(z). \end{aligned} \tag{4.13}$$

At $p = 1$, the operators $Y_i^{(N)}$ commute with symmetric Laurent polynomials considered as multiplication operators on $\mathbb{C}[z_1^{\pm 1}, \ldots, z_N^{\pm 1}]$. Hence we have

$$Y_i^{(N)} e_{-k}\tilde{\Phi}^{\lambda}_{\varsigma}(z) = e_{-k} Y_i^{(N)} \tilde{\Phi}^{\lambda}_{\varsigma}(z) = q^{2\varsigma(i)-N-1} e_{-k}\tilde{\Phi}^{\lambda}_{\varsigma}(z). \tag{4.14}$$

The ordered set of eigenvalues $\{q^{2\sigma(i)-N-1}\}_{i=1}^{N}$ determines the element $\sigma \in S^{\tilde{\lambda}}$ uniquely. Hence (4.13) and (4.14) lead to

$$e_{-k}\tilde{\Phi}^{\lambda}_{\varsigma}(z) = \tilde{\Phi}^{\tilde{\lambda}}_{\varsigma}(z) + \sum_{\mu \prec \tilde{\lambda}} c''_{\mu} \tilde{\Phi}^{\mu}_{\varsigma}(z). \tag{4.15}$$

Now let us consider any μ which appears in the sum (4.15). If there exists $i (< N - k)$ such that $\mu_i < \lambda_i$ then for $j > i$ we necessarily have $\mu_j < \lambda_j$ because of the assumption $\lambda_i - \lambda_{i+1} = 0$ or 1 and the fact that $\lambda_i < \lambda_j$ implies $\mu_i < \mu_j$, which follows since $\varsigma \in S^{\mu}$. However, $\mu_j < \lambda_j (j > i)$ leads to $|\tilde{\lambda}| > |\mu|$ which is in contradiction with $\mu \prec \tilde{\lambda}$. Thus we have $\mu_i \geqslant \lambda_i (i < N - k)$. If $\mu_i > \lambda_i$ for some i, then necessarily $\mu \not\preceq \tilde{\lambda}$ which is again a contradiction. Hence we obtain $\mu_i = \lambda_i (i < N - k)$. By the condition $\mu \prec \tilde{\lambda}$, we have $\mu_{N-k} = \tilde{\lambda}_{N-k}$. Because of the assumption on λ and the fact that $\lambda_i < \lambda_j$ implies $\mu_i < \mu_j$, we obtain $\mu_j \leqslant \tilde{\lambda}_j (j > N - k)$. Combining this with $|\tilde{\lambda}| = |\mu|$ we conclude that $\mu_j = \tilde{\lambda}_j \forall j$. That is the second term in the r.h.s. of (4.15) is zero. □

4.2. The decomposition

Let us consider the quotient space F_M/H'_-F_M and for each $k \geqslant 0$ its subspace $F_M^k/(H'_-F_M \cap F_M^k)$.

It is straightforward to establish the necessary and sufficient condition for the vector $\omega = \sum_\lambda \sum_{\sigma \in S^\lambda} \tilde{\Phi}_\sigma^\lambda(z) \otimes \psi_\sigma^\lambda (\in \mathbb{C}[z_1^{\pm 1}, \dots, z_N^{\pm 1}] \otimes (\otimes^N V))$ to be equivalent to 0 in the quotient space $\wedge^N V(z)$. The result is

$$\forall \lambda \begin{cases} \psi^\lambda_{\sigma(i,i+1)} = -\check{R}_{i,i+1}(q^{2(\sigma(i)-\sigma(i+1))})\psi^\lambda_\sigma & \forall \sigma \text{ s.t. } \lambda_{\sigma(i)} > \lambda_{\sigma(i+1)} \\ (q^{-2}S^{-1}_{i,i+1} - S_{i,i+1})\psi^\lambda_\sigma = 0 & \forall \sigma \text{ s.t. } \lambda_{\sigma(i)} = \lambda_{\sigma(i+1)} \end{cases} \tag{4.16}$$

where $\check{R}_{i,i+1}(x)$ is defined in (3.8). In view of the properties of the set S^λ, in the space $\wedge^N V(z)$ we have

$$\tilde{\Phi}^\lambda_\sigma(z) \otimes \psi_\sigma \sim \tilde{\Phi}^\lambda_{\min}(z) \otimes \check{R}_{i_r,i_r+1}(q^{2(\sigma_r(i_r+1)-\sigma_r(i_r))}) \dots \check{R}_{i_1,i_1+1}(q^{2(\sigma_1(i_1+1)-\sigma_1(i_1))})\psi_\sigma. \tag{4.17}$$

Here we used the notations of section 4.1.

By the triangularity of the non-symmetric Macdonald polynomial (4.4) and the relation (4.17), we obtain

$$\begin{aligned} V_M^{s+nk,k} &= \bigoplus_\lambda (E^\lambda \otimes (\otimes^N V))/\Omega \cap (E^\lambda \otimes (\otimes^N V)) \\ &= \bigoplus_\lambda (\tilde{\Phi}^\lambda_{\min}(z) \otimes (\otimes^N V))/\Omega \cap (\tilde{\Phi}^\lambda_{\min}(z) \otimes (\otimes^N V)) \end{aligned} \tag{4.18}$$

where the summation is over $\lambda \in \tilde{\mathcal{M}}^n_N$, such that $\lambda_1 \leqslant m^0_{s+nk}$, $|\boldsymbol{m}^0 - \lambda^{\min}| = k$.

Proposition 8. Define the set $\tilde{\mathcal{M}}^{n,k}_{s+nk}$ as

$$\tilde{\mathcal{M}}^{n,k}_{s+nk} = \{\lambda \in \tilde{\mathcal{M}}^n_{s+nk} | \lambda_1 \leqslant m^0_{s+nk}, |\boldsymbol{m}^0 - \lambda^{\min}| = k \text{ and } \lambda_i - \lambda_{i+1} = 0 \text{ or } 1\}. \tag{4.19}$$

Every vector from the linear space $F_M^k/(H'_-F_M \cap F_M^k)$ can be expressed as a linear combination of vectors of the form $\wedge(\tilde{\Phi}^\lambda_{\min}(z) \otimes \psi^\lambda)|M - s - nk\rangle$, where $\lambda \in \tilde{\mathcal{M}}^{n,k}_{s+nk}$ and $\psi^\lambda \in \otimes^N V$.

Proof. By the equation (4.18) it is sufficient to show that $\wedge(\tilde{\Phi}^\lambda_{\min}(z) \otimes \psi^\lambda)|M - s - nk\rangle (\lambda in \tilde{\mathcal{M}}_{s+nk}, \lambda_1 \leqslant m^0_{s+nk}, |\boldsymbol{m}^0 - \lambda^{\min}| = k, \psi^\lambda \in \otimes^N V)$ is equivalent to 0 in the space $F_M^k/(H'_-F_M \cap F_M^k)$ unless $\lambda_i - \lambda_{i+1} = 0$ or 1 for all $i = 1, \dots, N-1$. We will prove this by induction with respect to the ordering of the set $\tilde{\mathcal{M}}_{s+nk}$. (Note that if λ is not n-strict then $\wedge(\tilde{\Phi}^\lambda_{\min}(z) \otimes \psi^\lambda) = 0$.)

Since $\lambda_i - \lambda_{i+1} \neq 0, 1$ implies that $(\lambda_1, \dots, \lambda_i - 1, \lambda_{i+1} + 1, \dots)$ is lower with respect to the ordering of $\tilde{\mathcal{M}}_{s+nk}$, the minimal element satisfies the condition of proposition 8.

Fix λ and assume that the proposition is proved for all μ such that $\mu \prec \lambda$. Define $\tilde{\lambda} \in \tilde{\mathcal{M}}_{s+nk}$ as follows:

$$\lambda_i = \lambda_{i+1} \Leftrightarrow \tilde{\lambda}_i = \tilde{\lambda}_{i+1} \tag{4.20}$$

$$\lambda_1 = \tilde{\lambda}_1 \tag{4.21}$$

$$\tilde{\lambda}_i \neq \tilde{\lambda}_{i+1} \Rightarrow \tilde{\lambda}_i = \tilde{\lambda}_{i+1} + 1. \tag{4.22}$$

For each positive integer l, define $n_l = \#\{i | \tilde{\lambda}_i - \lambda_i = l\}$. If $n_l = 0$ for all $l (\geqslant 1)$ then either $\wedge(\tilde{\Phi}^\lambda_{\min}(z) \otimes \psi^\lambda)|M - s - nk\rangle$ itself satisfies the condition of proposition 8, or else the element $\prod_{l\geqslant 1} B_{-l}^{n_l} \cdot \wedge(\tilde{\Phi}_{\min}^{\tilde{\lambda}}(z)\otimes\psi_\lambda)|M-s-nk\rangle$ is in $H'_-F_M \cap F_M^k$. Expanding the last element, in the space $F_M^k/(H'_-F_M\cap F_M^k)$ we obtain

$$\wedge(\tilde{\Phi}_{\min}^{\lambda}(z)\otimes\psi^\lambda)|M-s-nk\rangle + \sum_{\mu\prec\lambda,\sigma\in S^\mu} \wedge(\tilde{\Phi}_{\sigma}^{\mu}(z)\otimes\psi_\sigma^\mu)|M-s-nk\rangle \sim 0 \tag{4.23}$$

for some ψ_σ^μ. By (4.17) and the induction assumption the proposition is proven. □

Proposition 9. For each $\lambda\in\tilde{\mathcal{M}}_{s+nk}^{n,k}$, define J and r_j such that $\lambda_1=\cdots=\lambda_{r_J}>\lambda_{r_J+1}=\cdots=\lambda_{r_J+r_{J-1}}>\cdots\geqslant\lambda_{r_1+\cdots+r_J(=N)}$, then in the space $F_M^k/(H'_-F_M\cap F_M^k)$, for each $\psi\in\otimes^N V$ we have

$$\wedge(\tilde{\Phi}_{\min}^{\lambda}(z)\otimes R_{i,i+1}(q^2)\psi)|M-s-nk\rangle\sim 0 \qquad (\lambda_{\min(i)}=\lambda_{\min(i+1)}) \tag{4.24}$$

$$\wedge\left(\tilde{\Phi}_{\min}^{\lambda}(z)\otimes\prod_{\substack{1\leqslant a\leqslant r_j\\ 0\leqslant b\leqslant r_{j+1}-1}} R_{l_j+a,l_j+r_j+r_{j+1}-b}(q^{-2(a+b)})\psi\right)\Bigg|M-s-nk\rangle\sim 0 \tag{4.25}$$

where (a,b) is on the right to (a',b') in the product if $a<a'$ or ($a=a'$ and $b<b'$), $l_j=\sum_{i=1}^{j-1} r_j$ and $l_0=0$.

Proof. The first relation follows from (4.16) and the identity

$$\operatorname{Im}(q^2 S_{i,i+1}^{-1}-S_{i,i+1})=\operatorname{Ker}(q^{-2}S_{i,i+1}^{-1}-S_{i,i+1}). \tag{4.26}$$

Consider the second relation. We define

$$\bar{\lambda}=(\lambda_1,\ldots,\lambda_{r_{j+1}+\ldots+r_J},\lambda_{1+r_{j+1}+\ldots+r_J}+1,\ldots,\lambda_N+1). \tag{4.27}$$

By lemma 1, the definition of the space $F_M^k/(H'_-F_M\cap F_M^k)$ and the relation (6.51) in [13], we obtain

$$\begin{aligned} &f(B_{-1},\ldots,B_{-(r_1+\ldots+r_j)})\cdot\wedge(\tilde{\Phi}_{\min}^{\bar{\lambda}}(z)\otimes\psi)|M-s-nk\rangle\\ &\quad=(f(B_{-1},\ldots,B_{-(r_1+\ldots+r_j)})\cdot\wedge(\tilde{\Phi}_{\min}^{\bar{\lambda}}(z)\otimes\psi))|M-s-nk\rangle\\ &\quad=\wedge(\tilde{\Phi}_{\varsigma}^{\lambda}(z)\otimes\psi)|M-s-nk\rangle\sim 0. \end{aligned}\tag{4.28}$$

Here $f(x_1,\ldots,x_l)$ is a polynomial such that

$$f\left(\sum_{i=1}^N z_i,\sum_{i=1}^N z_i^2,\ldots,\sum_{i=1}^N z_i^l\right)=\sum_{i_1<\ldots<i_l} z_{i_1}\ldots z_{i_l} \tag{4.29}$$

and $\varsigma\in S^\lambda$ is the minimal element of $S^{\bar{\lambda}}$.

If we apply the formula (4.17), we obtain

$$\wedge(\tilde{\Phi}_{\min}^{\lambda}(z)\otimes\prod_{\substack{0\leqslant a\leqslant r_j-1\\ 0\leqslant b\leqslant r_{j+1}-1}}\check{R}_{l_j+r_{j+1}-b+a,l_j+r_{j+1}-b+a+1}(q^{-2(a+b+1)})\psi)|M-s-nk\rangle\sim 0 \tag{4.30}$$

where (a,b) on the right to (a',b') in the product if $a<a'$ or ($a=a'$ and $b<b'$). Finally, taking into account the relation

$$\begin{aligned}&\prod_{\substack{0\leqslant a\leqslant r_j-1\\ 0\leqslant b\leqslant r_{j+1}-1}}\check{R}_{l_j+r_{j+1}-b+a,l_j+r_{j+1}-b+a+1}(q^{-2(a+b+1)})\prod_{\substack{1\leqslant a\leqslant r_j\\ 0\leqslant b\leqslant r_{j+1}-1}}P_{l_j+a,l_j+r_j+r_{j+1}-b}\\ &\quad=\prod_{\substack{1\leqslant a\leqslant r_j\\ 0\leqslant b\leqslant r_{j+1}-1}}R_{l_j+a,l_j+r_j+r_{j+1}-b}(q^{-2(a+b)})\end{aligned}\tag{4.31}$$

we obtain (4.25). □

With notations of proposition 9, for each $\lambda \in \tilde{\mathcal{M}}^{n,k}_{s+nk}$ define the linear subspace of $\otimes^N V$

$$V^{\lambda} = \sum_{\lambda_{\min(i)}=\lambda_{\min(i+1)}} \mathrm{Im}\, R_{i,i+1}(q^2) + \sum_{j=1}^{J-1} \mathrm{Im}\left(\prod_{\substack{1\leqslant a\leqslant r_j \\ 1\leqslant b\leqslant r_{j+1}-1}} R_{l_j+a,l_j+r_j+r_{j+1}-b}(q^{-2(a+b)}) \right). \qquad (4.32)$$

By proposition 9, we obtain the following.

Proposition 10. Consider the map

$$\begin{aligned} \psi_k :\ & \bigoplus_{\lambda\in\tilde{\mathcal{M}}^{n,k}_{s+nk}} \tilde{\Phi}^{\lambda}_{\min}(z) \otimes (\otimes^{s+nk} V/V^{\lambda}) \to F^k_M/(H'_- F_M \cap F^k_M) \\ & v \mapsto v \wedge |M-s-nk\rangle. \end{aligned} \qquad (4.33)$$

Define the action of $U'_q(\widehat{\mathfrak{sl}}_n)$ on the l.h.s. of (4.33) by (2.22)–(2.27).

Then the map ψ_k is well defined, surjective and is a $U'_q(\widehat{\mathfrak{sl}}_n)$-intertwiner.

Proposition 11. Let $\lambda \in \tilde{\mathcal{M}}^{n,k}_{s+nk(=N)}$. For any such λ we define J, r_j and l_j in the same way as in proposition 9. Let θ be the border strip characterized by $\langle r_J, r_{J-1}, \ldots, r_1\rangle$. We have

$$\otimes^N V/V^{\lambda} = \otimes^N V/\,\mathrm{Ker}\, \bar{R}_{\theta}. \qquad (4.34)$$

Proof. First we will show that $\otimes^N V/V^{\lambda} \supset \otimes^N V/\,\mathrm{Ker}\, \bar{R}_{\theta}$. Applying repeatedly the Yang–Baxter equation $R_{a,b}(x)R_{a,c}(xy)R_{b,c}(y) = R_{b,c}(y)R_{a,c}(xy)R_{a,b}(x)$, we can move some special elements to the right in the product

$$\bar{R}_{\theta} = \ldots R_{i+1,i}(q^{-2}) = \ldots \prod_{\substack{1\leqslant a\leqslant r_{j+1} \\ 0\leqslant b\leqslant r_j-1}} R_{l_j+r_j+a,l_j+r_j-b}(q^{2(a+b)}) \qquad (4.35)$$

where (a,b) on the right to (a',b') in the product if $a < a'$ or ($a = a'$ and $b < b'$), and $\lambda_{\min(i)} = \lambda_{\min(i+1)}$. By the formula

$$R_{b,a}(x)R_{a,b}(x^{-1}) = \frac{(x-q^2)(x^{-1}q^{-2}-1)}{(x-1)(x^{-1}-1)} id \qquad (4.36)$$

we obtain $\otimes^N V/V^{\lambda} \supset \otimes^N V/\,\mathrm{Ker}\, \bar{R}_{\theta}$.

Next we will show that $\otimes^N V/V^{\lambda} \subset \otimes^N V/\,\mathrm{Ker}\, \bar{R}_{\theta}$. We show that the vectors $\otimes^N_{i=1} v_{e_i}$ such that $e_i < e_{i+1}$ if $\lambda_{\min(i)} = \lambda_{\min(i+1)}$, $e_i \geqslant e_{i+1}$ if $\lambda_{\min(i)} \neq \lambda_{\min(i+1)}$ span the space $\otimes^N V/V^{\lambda}$.

Using the relations $R_{i',i'+1}(q^2)(\otimes^N v_{e_i}) \sim 0(\lambda_{\min(i')} = \lambda_{\min(i'+1)})$ we find that the set of vectors $\{\otimes^N_{i=1} v_{e_i} |$ if $\lambda_{\min(i)} = \lambda_{\min(i+1)}$ then $e_i < e_{i+1}\}$ spans the space $\otimes^N V/V^{\lambda}$. For each vector of the form $\otimes^N_{i=1} v_{e_i}$ we define $\tilde{N}(\otimes^N_{i=1} v_{e_i}) = \#\{(i,j) | i < j, e_i \geqslant e_j$ and $\lambda_{\min(i)} \neq \lambda_{\min(j)}\}$. Consider the vector $\otimes^N_{i=1} v_{e_i}$ such that if $\lambda_{\min(i)} = \lambda_{\min(i+1)}$ then $e_i < e_{i+1}$. Assume that there is an i such that $e_i < e_{i+1}$ (if $\lambda_{\min(i)} \neq \lambda_{\min(i+1)}$), then we obtain the relation

$$\prod_{\substack{1\leqslant a\leqslant r_j \\ 1\leqslant b\leqslant r_{j+1}-1}} R_{l_j+a,l_j+r_j+r_{j+1}-b}(q^{-2(a+b)})(\otimes^N v_{e_i}) \sim 0 \qquad (4.37)$$

for all possible j. By (4.37), the vector $\otimes^N v_{e_i}$ is equivalent to a linear combination of the vectors $\otimes^N v_{e_{i'}}$ such that $\tilde{N}(\otimes^N v_{e_{i'}}) < \tilde{N}(\otimes^N v_{e_i})$. Because $\otimes^N v_{e_{i'}}$ is invariant by the relations $R_{i',i'+1}(q^2)(\otimes^N v_{e_i})$ $(l_j+1 \leqslant i' \leqslant l_{j+1}-1, l_{j+1}+1 \leqslant i' \leqslant l_{j+2}-1)$, if we use these relations we obtain that the vector $\otimes^N v_{e_i}$ ($e_i < e_{i+1}$ if $\lambda_{\min(i)} = \lambda_{\min(i+1)}$) is expressed by the sum of $\otimes^N v_{e_{i'}}$ such that $\tilde{N}(\otimes^N v_{e_{i'}}) < \tilde{N}(\otimes^N v_{e_i})$ and $e_{i'} < e_{i'+1}$ (if $\lambda_{\min(i')} \neq \lambda_{\min(i'+1)}$).

By the induction on $\tilde{N}$ we find that the vectors $\otimes^N v_{e_i}$ ($e_i < e_{i+1}$ if $\lambda_{\min(i)} = \lambda_{\min(i+1)}$, $e_i \geqslant e_{i+1}$ if $\lambda_{\min(i)} \neq \lambda_{\min(i+1)}$) span the space $\otimes^N V/V^\lambda$.

The number of these vectors is equal to the number of SST of θ. By corollary 1, we obtain $\otimes^N V/V^\lambda \subset \otimes^N V/\operatorname{Ker}\bar{R}_\theta$. □

In what follows we identify the border strips $\langle m_1, \ldots, m_r\rangle$ and $\langle m_1, \ldots, m_r, \overbrace{n, \ldots, n}^{}\rangle$, and identify $(\lambda_1, \ldots, \lambda_{s+nk})$ and $(\overbrace{\lambda_1 + 1, \ldots, \lambda_1 + 1}^{n}, \lambda_1, \ldots, \lambda_{s+nk})$ which are elements of $\coprod_k \tilde{\mathcal{M}}^{n,k}_{s+nk}$. Proposition 11 gives a one-to-one correspondence of $\coprod_k \tilde{\mathcal{M}}^{n,k}_{s+nk}$ and the set of all skew Young diagrams of the border-strip type $\langle m_1, \ldots, m_r\rangle$ which satisfy $m_i \leqslant n$ for all i and $\sum_{i=1}^r m_i \equiv s \bmod n$. On this correspondence the degree of the semi-infinite wedge $\wedge(\tilde{\Phi}^\lambda_{\min}(z) \otimes \psi)|M - s - nk\rangle$ is equal to $\frac{1-n}{24} - \frac{k(n-k)}{2n} + \frac{1}{2n}|\theta|(n - |\theta|) + t(\theta)$, where θ is the border strip which corresponds to λ and $t(\theta) = \sum_{i=1}^{r-1}(r-i)m_i$.

We define $\operatorname{ch}(F_M/H'_- F_M) = \sum_{\mu,i} \dim(V_{\mu,i}) e^\mu q^i$, where $V_{\mu,i}$ is the subspace of $F_M/H'_- F_M$ of the degree (2.37) i and of the $U_q(\mathfrak{sl}_n)$-weight μ. We put $\sum_{\mu,i} a_{\mu,i} e^\mu q^i \leqslant \sum_{\mu,i} b_{\mu,i} e^\mu q^i$ iff $a_{\mu,i} \leqslant b_{\mu,i}$ for all μ and i. By proposition 10 we have

$$\operatorname{ch}(F_M/H'_- F_M) \leqslant q^{\frac{1-n}{24} - \frac{k(n-k)}{2n}} \sum_{\substack{\theta \in BS \\ |\theta| \equiv k \bmod n}} q^{\frac{1}{2n}|\theta|(n-|\theta|) + t(\theta)} s_\theta(z). \tag{4.38}$$

$F_M/H'_- F_M$ with U_1-action is isomorphic to $V(\Lambda_k)$, this isomorphism is degree preserving with respect to the degree (2.37) on $F_M/H'_- F_M$ and the homogeneous degree on $V(\Lambda_k)$ and the character formula of $V(\Lambda_k)$ is given in proposition 7. Hence the inequality of (4.38) is, in fact, an equality and, therefore, the map (4.33) must is bijective. Thus have the following theorem.

Theorem 12. We have the isomorphism of $U'_q(\widehat{\mathfrak{sl}}_n)$-modules:

$$F_M/H'_- F_M \simeq \bigoplus_\theta V_\theta \tag{4.39}$$

where the sum is over all border strips $\langle m_1, \ldots m_r\rangle$, ($m_i \leqslant n$, $m_r < n$ and $N \equiv M$ mod n), the space V_θ and the level-0 $U'_q(\widehat{\mathfrak{sl}}_n)$-action is defined by $(\pi^{(N)}_{a_1,\ldots,a_N}, R_\theta \cdot \otimes^N V)$ where $N = \sum_{i=1}^r m_i$ and $a_{l + \sum_{i=j}^r m_i} = 2(l - 1 + \sum_{i=1}^{j-1} m_i)$.

4.3. $\mathfrak{sl}_2$ *case*

In this section we will discuss the $\mathfrak{sl}_2$ case in a somewhat more detail.

Let W_n be the $(n+1)$-dimensional irreducible module of $U_q(\mathfrak{sl}_2)$, and $W_n(b)$ be the evaluation module with the parameter b whose $U'_q(\widehat{\mathfrak{sl}}_2)$-module structure is given by

$$E_0 = q^b F_1 \qquad F_0 = q^{-b} E_1 \qquad K_0 = K_1^{-1}. \tag{4.40}$$

It is known that every finite-dimensional irreducible $U'_q(\widehat{\mathfrak{sl}}_2)$-module is isomorphic to $\otimes_\mu W_{n_\mu}(b_\mu)$ for some n_μ and b_μ. We will represent the $U'_q(\widehat{\mathfrak{sl}}_2)$-module described by a skew Young diagram as the tensor product of the form $\otimes_\mu W_{n_\mu}(b_\mu)$.

Proposition 13. Let θ be the skew Young diagram of border strip $\langle m_1, \ldots, m_r\rangle$ such that $m_i = 1$ or 2, and $N = \sum_{i=1}^r m_i$, $a_{l+\sum_{i=j}^r m_i} = 2(l - 1 + \sum_{i=1}^{j-1} m_i)$. We put $I = \{i | m_i = 1$

and $m_{i-1} = 2\} = \{l_1, l_2, \ldots, l_{r'}\}(m_0 = 2, l_i < l_{i+1})$ and let n_i be the integer such that $m_{l_i} = \ldots = m_{l_i+n_i-1} = 1, m_{l_i+n_i} = 2$ and $b_i = 2l_i + n_i - 3$.

The $U'_q(\widehat{\mathfrak{sl}}_2)$-module $(\pi^{(N)}_{a_1,\ldots,a_N}, R_\theta \cdot \otimes^N \mathbb{C}^2)$ is isomorphic to $W_{n_1}(b_1) \otimes W_{n_2}(b_2) \otimes \cdots \otimes W_{n_{r'}}(b_{r'})$.

Proof. By proposition 5, we find that $(\pi^{(N)}_{a_1,\ldots,a_N}, R_\theta \cdot \otimes^N \mathbb{C}^2)$ is isomorphic to $(\pi^{(N)}_{a_N,\ldots,a_1}, \otimes^N \mathbb{C}^2 / \operatorname{Ker} \bar{R}_\theta)$.

As in the proof of proposition 11, we obtain $\operatorname{Im} \check{R}_{i,i+1}(q^2) \subset \operatorname{Ker} \bar{R}_\theta$ if $a_{i+1} = a_i - 2$ and $\operatorname{Im} \check{R}_{i,i+1}(q^{-2}) \subset \operatorname{Ker} \bar{R}_\theta$ if $a_{i+1} = a_i + 2$.

We can directly confirm that the $U'_q(\widehat{\mathfrak{sl}}_2)$-module $(\pi_{a,a-2}, \mathbb{C}^2 \otimes \mathbb{C}^2 / \operatorname{Im} \check{R}_{1,2}(q^2))$ is one-dimensional and the module $(\pi_{a,a+2,\ldots,a+2(l-1)}, \otimes^l \mathbb{C}^2 / \sum_{i=1}^{l-1} \operatorname{Im} \check{R}_{i,i+1}(q^{-2}))$ is isomorphic to $W_l(a + l - 1)$, where (l) is the Young diagram of degree l which has only one row.

If we put

$$\tilde{V} = \sum_{i|a_{i+1}=a_i+2} \operatorname{Im} \check{R}_{i,i+1}(q^2) + \sum_{i|a_{i+1}=a_i-2} \operatorname{Im} \check{R}_{i,i+1}(q^{-2}) \tag{4.41}$$

then $(\pi^{(N)}_{a_N,\ldots,a_1}, \otimes^N \mathbb{C}^2 / \tilde{V}) \simeq W_{n_1}(b_1) \otimes W_{n_2}(b_2) \otimes \cdots \otimes W_{n_{r'}}(b_{r'})$.

Since the dimension of $W_{n_1}(b_1) \otimes W_{n_2}(b_2) \otimes \cdots \otimes W_{n_{r'}}(b_{r'})$ is equal to the dimension $\otimes^N \mathbb{C}^2 / \operatorname{Ker} \bar{R}_\theta$ the proof is finished. □

By proposition 13, we can rewrite the decomposition (4.39) for $\mathfrak{sl}_2$ case. In fact we obtain the same decomposition as [8]. More precisely we obtain:

Proposition 14. If we change the coproduct of the level-0 $U'_q(\widehat{\mathfrak{sl}}_2)$-module defined in [8] to fit our coproduct (2.11)–(2.13), the level-0 $U'_q(\widehat{\mathfrak{sl}}_2)$-module of $V(\Lambda_s)$ ($s \equiv M \bmod 2$, $s = 0$ or 1) defined in [8] is isomorphic to the level-0 $U'_q(\widehat{\mathfrak{sl}}_2)$-module of $V(\Lambda_s)$ defined in this paper, and the degree is preserved under this isomorphism.

Acknowledgments

I am grateful to D Uglov with whom we collaborated on the papers [16, 13]. I also thank M Jimbo, M Kashiwara, T Miwa, M Nazarov, Y Saito, V Tarasov and colleagues at RIMS for discussions and support. KT was supported by the JSPS Research Fellowship for Young Scientist.

References

[1] Akasaka T and Kashiwara M 1998 Finite-dimensional representations of quantum affine algebra *Publ. Res. Inst. Math. Sci.* to appear

[2] Bernard D, Pasquier V and Serban D 1994 Spinons in conformal field theory *Nucl. Phys.* B **428** 612–28

[3] Cherednik I V 1987 A new interpretation of Gelfand–Tzetlin bases *Duke Math. J.* **54** 563–77

[4] Cherednik I V 1995 Double affine Hecke algebras and Macdonald's conjectures *Ann. Math.* **141** 191–216

[5] Ginzburg V, Kapranov M and Vasserot E 1995 Langlands reciprocity for algebraic surfaces *Math. Res. Lett.* **2** 147–60

[6] Hayashi T 1990 Q-analogues of Clifford and Weyl algebras—spinor and oscillator representations of quantum enveloping algebras *Commun. Math. Phys.* **127** 129–44

[7] Haldane F D M, Ha Z N C, Talstra J C, Bernard D and Pasquier V 1992 Yangian symmetry of integrable quantum chains with long-range interactions and a new description of states in conformal field theory *Phys. Rev. Lett.* **69** 2021–5

[8] Jimbo M, Kedem R, Konno H, Miwa T and Petersen J-U H 1995 Level-0 action of $U_q(\widehat{\mathfrak{sl}}_2)$ on level-1 modules and Macdonald polynomials *J. Phys. A: Math. Gen.* **28** 5589

[9] Kashiwara M, Miwa T and Stern E 1995 Decomposition of q-deformed Fock Spaces *Selecta Mathematica* **1** 787–805

[10] Kirillov A N, Kuniba A and Nakanishi T 1997 Skew Young diagram method in spectral decomposition of integrable lattice models *Commun. Math. Phys.* **185** 441–65

[11] Knop F 1997 Integrality of two variable Kostka functions *Preprint*

[12] Macdonald I G 1995 *Symmetric Functions and Hall Polynomials* 2nd edn (New York: Clarendon)

[13] Saito Y, Takemura K and Uglov D 1998 Toroidal actions on level 1 modules of $U_q(\widehat{\mathfrak{sl}}_n)$ *Transformation Groups* to appear

[14] Schoutens K 1994 Yangian symmetry in conformal field theory *Phys. Lett.* B **331** 335–41

[15] Stern E 1995 Semi-infinite wedges and vertex operators *Int. Math. Res. Notices* **4** 201–20

[16] Takemura K and Uglov D 1998 Level-0 action of $U_q(\widehat{\mathfrak{sl}}_n)$ on the q-deformed Fock spaces *Commun. Math. Phys.* to appear

[17] Uglov D 1997 The trigonometric counterpart of the Haldane–Shastry model *Preprint* RIMS-1030 (hep-th/9508145)

[18] Varagnolo M and Vasserot E 1996 Schur duality in the toroidal setting *Commun. Math. Phys.* **182** 469–84

[19] Varagnolo M and Vasserot E Double-loop algebras and the Fock space *Preprint* (q-alg/9612035)